

32nd United States of America Mathematical Olympiad

Day I 12:30 PM - 5 PM

April 29, 2003

1. Prove that for every positive integer n there exists an n -digit number divisible by 5^n all of whose digits are odd.
2. A convex polygon \mathcal{P} in the plane is dissected into smaller convex polygons by drawing all of its diagonals. The lengths of all sides and all diagonals of the polygon \mathcal{P} are rational numbers. Prove that the lengths of all sides of all polygons in the dissection are also rational numbers.
3. Let $n \neq 0$. For every sequence of integers

$$A = a_0, a_1, a_2, \dots, a_n$$

satisfying $0 \leq a_i \leq i$, for $i = 0, \dots, n$, define another sequence

$$t(A) = t(a_0), t(a_1), t(a_2), \dots, t(a_n)$$

by setting $t(a_i)$ to be the number of terms in the sequence A that precede the term a_i and are different from a_i . Show that, starting from any sequence A as above, fewer than n applications of the transformation t lead to a sequence B such that $t(B) = B$.

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4. Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E , respectively. Lines AB and DE intersect at F while lines BD and CF intersect at M . Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

5. Let a, b, c be positive real numbers. Prove that

$$\frac{(2a + b + c)^2}{2a^2 + (b + c)^2} + \frac{(2b + c + a)^2}{2b^2 + (c + a)^2} + \frac{(2c + a + b)^2}{2c^2 + (a + b)^2} \leq 8.$$

6. At the vertices of a regular hexagon are written six nonnegative integers whose sum is 2003. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.